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Institute of Mathematical Sciences
Division of Electromagnetic Research

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Abstract

The problem of diffraction of a plane wave by a circular aperture in a plane screen has been treated by Levine and Schwinger, who reduced it to a system of infinitely many linear equations. This system has been modified by Bouwkamp, and Magnus has proved that Bouwkamp's system is equivalent to a Fredholm integral equation of the second kind. If $\alpha = ka$, where k is the wave number of the incident wave and a denotes the radius of the circular aperture, then the coefficients of Bowkamp's sytem depend on α. Their asymptotic behavior for $\alpha \to \infty$ is investigated and explicit expressions for the first few terms are stated in this report. The integral equation equivalent to Bouwkamp's system degenerates into an integral equation of the first kind as $\alpha \to \infty$. The solution of the degenerate integral equation can be given explicitly, and a perturbation method yields some formulas for the solution of the original integral equation if α is large.

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1. Introduction

We consider here the problem of the diffraction of a plane scalar wave normally incident on a plane screen with a circular aperture. Let u be the solution of $\Delta u + K^2 u = 0$, which represents the diffracted wave, and let ρ, θ be polar coordinates on the screen such that $\rho = 0$ represents the center of the aperture. We assume that on the screen $u = 0$. Let the radius of the aperture be a and let $\Phi(\rho)$ be the value of u in the aperture, $0 \leq \rho \leq a$. Bouwkamp [1], using the Levine and Schwinger [2] variational method, derived the following result.

If $\Phi(\rho)$ is expanded in a series of Legendre polynomials of the type

(1) $$\Phi(\rho) = \sum_{n=0}^{\infty} b_n P_{2n+1}\left(\sqrt{1 - \rho^2/a^2}\right)$$

then the b_n satisfy the system of infinitely many linear equations

(2) $$\sum_{n=0}^{\infty} d_{m,n} b_n = \frac{6}{i\alpha} \delta_{m,o} ,$$

where $\alpha = Ka$, $\delta_{o,o} = 1$ and $\delta_{m,o} = 0$ if $m \neq 0$, and where

(3) $$d_{m,n} = \left(\frac{6}{\alpha}\right)^2 \frac{\Gamma(n + \frac{3}{2})}{n!} \frac{\Gamma(m + \frac{3}{2})}{m!} g^*_{m,n}(\alpha)$$

and

(4) $$g^*_{m,n}(\alpha) = \int_0^{\infty} \frac{\sqrt{v^2 - 1}}{v^2} J_{2m+3/2}(\alpha v) J_{2m+3/2}(\alpha v)\, dv.$$

In (4), J denotes the Bessel function of the first kind, and

(5) $$\sqrt{v^2 - 1} = - i \sqrt{1 - v^2} , \qquad \sqrt{1 - v^2} > 0$$

for $0 \leq v < 1$. The diffracted amplitude in the forward direction is given by

(6) $$A_1 = \frac{-i\alpha}{3} b_o .$$

Instead of using Bouwkamp's coefficients b_n for the expansion of the field in the aperture, we introduce the quantities

(7) $$s_n = (-1)^n \frac{\Gamma(n + \frac{3}{2})}{n!} b_n .$$

Let $y = \sqrt{1 - \rho^2/a^2}$, so that

(8) $$\mathcal{E}(y) = \sum_{n=0}^{\infty} s_n P_{2n+1}(y) .$$

Then Magnus has shown [3] that $\mathcal{E}(y)$ satisfies the following integral equation:

(9) $$\frac{-i\alpha y}{\sqrt{\pi}} = \mathcal{E}(y) + \frac{2\alpha}{\pi} \int_0^1 G(x,y;\alpha)\, \mathcal{E}(x)\, dx, \qquad 0 \leq y \leq 1 .$$

The kernel $G(x,y)$ is given by

(10) $$G(x,y;\alpha) = -\frac{\pi}{4} \int_{|x-y|}^{x+y} \frac{J_1(\alpha\tau) + iH_1(\alpha\tau)}{\tau}\, d\tau ,$$

where J_1 is a Bessel function and H_1 is a Struve function and α is a real parameter.

From a knowledge of $\mathcal{E}(y)$ we can (using (8), (7) and (1)) obtain $\Phi(\rho)$, the value of u in the aperture.

2. The field in the aperture for $\alpha \to \infty$

Magnus has shown [3] that a solution of (9) exists for all real positive α. We wish to study the asymptotic solution of this equation for large postive α. Our first task is to determine how the solution $\mathcal{E}(x;\alpha)$ behaves for $\alpha \to \infty$. To do this we will need to know

$$\lim_{\alpha \to \infty} G(x,y;\alpha) \equiv G(x,y;\infty)$$

for $0 \leq x,\ y \leq 1$ and for $x \neq y$. From (10)

$$G(x,y;\infty) = \lim_{\alpha \to \infty} -\frac{\pi}{4} \int_{|x-y|}^{x+y} \frac{J_1(\alpha\tau) + iH_1(\alpha\tau)}{\tau}\, d\tau \ .$$

If $x \neq y$,

$$\lim_{\alpha \to \infty} -\frac{\pi}{4} \int_{|x-y|}^{x+y} \frac{J_1(\alpha\tau)}{\tau}\, d\tau = \lim_{\alpha \to \infty} -\frac{\pi}{4} J_1(\alpha\bar{\tau}) \int_{|x-y|}^{x+y} \frac{d\tau}{\tau}$$

$$= -\frac{\pi}{4} \log \left|\frac{x+y}{x-y}\right| \lim_{\alpha \to \infty} J_1(\alpha\bar{\tau}) \ ,$$

where $|x-y| \leq \bar{\tau}(\alpha) \leq |x+y|$. Since $x \neq y$, $J_1(\alpha\bar{\tau}) \to 0$ as $\alpha \to \infty$. Likewise

$$\lim_{\alpha \to \infty} -\frac{\pi}{4} \int_{|x-y|}^{x+y} \frac{H_1(\alpha\tau)}{\tau}\, d\tau = -\frac{\pi}{4} \log \left|\frac{x+y}{x-y}\right| \lim_{\alpha \to \infty} H_1(\alpha\tau^*) \ ,$$

where $|x-y| \leq \tau^*(\alpha) \leq |x+y|$. Since $x \neq y$,

$$\lim_{\alpha \to \infty} H_1(\alpha\tau^*) = \frac{1}{\Gamma(\frac{3}{2})\ \Gamma(\frac{1}{2})} = \frac{2}{\pi} \ .$$

Hence if $x \neq y$

(10a) $G(x,y;\infty) = -\frac{i}{2} \log \left|\frac{x+y}{x-y}\right|$, for $0 \leq x,\ y \leq 1$.

Let $\mathcal{E}_0(x)$ denote the limit of $\mathcal{E}(x;\alpha)$ as $\alpha \to \infty$. We can now show

<u>Theorem 1</u>. For $\alpha \to \infty$, the solution of (9) tends towards

$$\mathcal{E}_0(x) = \frac{1}{\sqrt{\pi}} \frac{x}{\sqrt{1-x^2}} \ .$$

Proof: Divide (9) by α and let $\alpha \to \infty$. We assume that $\lim\limits_{\alpha\to\infty} \dfrac{\mathcal{E}(y;\alpha)}{\alpha} = 0$, and that

$$\lim_{\alpha\to\infty} \int_0^1 G(x,y;\alpha)\ \mathcal{E}(x,\alpha)\,dx = \int_0^1 G(x,y;\infty)\ \mathcal{E}_0(x)\ dx\ ;$$

the validity of these assumptions will be justified later. Then, taking into account (10a), we have the following integral equation of the first kind for the determination of $\mathcal{E}_0(x)$:

$$y = \frac{1}{\sqrt{\pi}} \int_0^1 \log\left|\frac{x+y}{x-y}\right| \mathcal{E}_0(x)\ dx; \qquad 0 \le y \le 1\ . \tag{11}$$

The easiest way to obtain the solution of (11) is to proceed from the known solution of the following integral equation [4]:

$$y = -\frac{1}{\pi} \int_{-1}^{+1} \frac{x}{\sqrt{1-x^2}}\ \log|x-y|\ dx\ ; \qquad -1 \le y \le 1\ . \tag{12}$$

This may be transformed as follows:

$$\begin{aligned} y &= -\frac{1}{\pi}\int_{-1}^{0} \frac{x}{\sqrt{1-x^2}}\ \log|x-y|dx - \frac{1}{\pi}\int_0^{+1} \frac{x}{\sqrt{1-x^2}}\ \log|x-y|dx \\ &= \frac{1}{\pi}\int_0^1 \frac{x}{\sqrt{1-x^2}}\ \log|x+y|dx - \frac{1}{\pi}\int_0^1 \frac{x}{\sqrt{1-x^2}}\ \log|x-y|dx \\ &= \frac{1}{\pi}\int_0^1 \frac{x}{\sqrt{1-x^2}}\ \log\left|\frac{x+y}{x-y}\right|\ dx\ . \end{aligned}$$

Comparison of this result with equation (11) yields the desired result

$$\mathcal{E}_0(x) = \frac{1}{\sqrt{\pi}}\ \frac{x}{\sqrt{1-x^2}}\ .$$

For this limiting case we have the first transmission coefficient A_1:

$$A_1 = -\frac{i\alpha}{3} b_o = -\frac{i\alpha}{3} \frac{s_o}{\Gamma(\frac{3}{2})} = \frac{-i\alpha}{3\Gamma(\frac{3}{2})} \cdot 3 \int_o^1 P_1(x)\,\mathcal{E}_o(x)\,dx$$

$$= \frac{-2i\alpha}{\sqrt{\pi}} \int_o^1 x \frac{1}{\sqrt{\pi}} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{i\alpha}{2} \quad .$$

We now justify our assumption that letting $\alpha \to \infty$ while keeping the radius of the aperture constant is equivalent to keeping α constant while letting the radius of the aperture approach ∞. If α is constant and the radius of the aperture tends to infinity we know that the wave goes through undisturbed. Hence if $\mathcal{E}_o(x)$ makes $\Phi(\rho)$ constant in the aperture we feel justified in our assumption. We will now show that this is indeed the case.

We have from (8)

$$\mathcal{E}_o(x) = \frac{1}{\sqrt{\pi}} \frac{x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} s_n P_{2n+1}(x) \quad . \tag{13}$$

Using the orthogonality property of the Legendre polynomials we get

$$s_n = \frac{4n+3}{\sqrt{\pi}} \int_o^1 \frac{x}{\sqrt{1-x^2}} P_{2n+1}(x)\,dx$$

$$= \frac{4n+3}{4\sqrt{\pi}} \int_o^{2\pi} P_{2n+1}(\cos\theta) \cos\theta\,d\theta$$

$$= \frac{4n+3}{4\sqrt{\pi}} \cdot 2\pi \binom{2n}{n} \binom{2n+2}{n+1} 2^{-4n-2} \qquad {}^*$$

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* See [4], p. 51.

$$= \frac{4n+3}{2} \sqrt{\pi} \; \frac{\Gamma(2n+1)\,\Gamma(2n+3)}{n!\,n!\,(n+1)!\,(n+1)!\; 2^{4n+2}}$$

$$= \frac{4n+3}{2} \sqrt{\pi} \; \frac{\Gamma(n+\frac{1}{2})\, n!\; 2^{2n+1}\; \Gamma(n+\frac{3}{2})\,(n+1)!\; 2^{2n+3}}{(n!)^2 (n+1)!\,(n+1)!\; 2^{4n+2}\; 2\sqrt{\pi} \cdot 2\sqrt{\pi}} \quad \dagger$$

$$= \frac{4n+3}{2\sqrt{\pi}} \; \frac{\Gamma(n+\frac{1}{2})\;\Gamma(n+\frac{3}{2})}{n!\;(n+1)!} \; .$$

From (7) we get

$$(14) \qquad b_n = \frac{(-1)^n n!}{\Gamma(n+\frac{3}{2})} \; \frac{4n+3}{2\sqrt{\pi}} \; \frac{\Gamma(n+\frac{1}{2})\;\Gamma(n+\frac{3}{2})}{n!\;(n+1)!}$$

$$= (-1)^n \; \frac{4n+3}{2\sqrt{\pi}} \; \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+2)} \; .$$

Let us now determine the C_n in the relation

$$1 = \sum_{n=0}^{\infty} C_n \, P_{2n+1}(x) \; .$$

We get

$$C_n = \frac{1}{4n+3} \int_0^1 1 \cdot P_{2n+1}(x)\, dx = \frac{4n+3}{2n+1} (-1)^n \; \frac{\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n+1}{2}}{(n+1)!}$$

$$= (-1)^n \; \frac{4n+3}{2\sqrt{\pi}} \; \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+2)} \; .$$

Hence $C_n = b_n$ and $\Phi(\rho)$ is constant in the aperture.

$\dagger$ Here we have used the formula $\Gamma(2z) = \dfrac{1}{2\sqrt{\pi}} \, 2^{2z} \, \Gamma(z) \, \Gamma(z+\frac{1}{2})$.

There exists a general formula which permits us to express the field $\Phi(\rho)$ in the aperture in terms of the solution $\mathcal{E}(y)$ of the integral equation (9). The connection between the two functions is given by the following:

Lemma 1: Let $\mathcal{E}(y)$ be given by (8) and let

$$\Phi(x) = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{\Gamma(n+\frac{3}{2})} s_n P_{2n+1}(x) .$$

Then

$$\Phi(x) = \int_0^1 \Omega(x,y)\, \mathcal{E}(y)\, dy ,$$

where

$$\Omega(x,y) = \frac{1}{i\sqrt{\pi}} \int_0^1 \left\{ W(x,y,it) - W(x,y,-it) \right\} \frac{dt}{\sqrt{1-t^2}}$$

and

$$W(x,y,t) = \frac{1}{2\pi} \int_0^{\pi} \frac{(1-t^2)\, d\omega}{\left\{1-2t\left[xy+(1-x^2)^{1/2}(1-y^2)^{1/2}\cos\omega\right] + t^2\right\}^{3/2}} .$$

Proof: According to Watson [5],

$$W(x,y,t) = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) P_n(x) P_n(y) t^n .$$

Therefore,

$$\frac{1}{2i} \left\{ W(x,y,it) - W(x,y,-it) \right\} = \sum_{n=0}^{\infty} (2n+\tfrac{3}{2}) P_{2n+1}(x) P_{2n+1}(y) (-1)^n t^{2n+1} .$$

Since

$$\int_0^1 t^{2n+1} \frac{dt}{\sqrt{1-t^2}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} ,$$

we see that

$$\Omega(x,y) = \sum (-1)^n \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} P_{2n+1}(x) P_{2n+1}(y).$$

This and the orthogonality relations for the Legendre functions prove Lemma 1.

3. Asymptotic expansion of the diffraction integrals

In this section we study the asymptotic expansion of the diffraction integrals (4). (4) may be written

$$g^*_{m,n}(a) = -i\int_0^1 \frac{\sqrt{1-v^2}}{v^2} J_{2m+3/2}(av)J_{2n+3/2}(av)dv + \int_1^\infty \frac{\sqrt{v^2-1}}{v^2} J_{2m+3/2}(av)J_{2n+3/2}(av)dv.$$

We treat the real and imaginary parts separately. Consider

$$-\operatorname{Im} g^*_{m,n}(a) = \int_0^1 \frac{\sqrt{1-v^2}}{v^2} J_\mu(av)\, J_\nu(av)\, dv$$

where $\mu = 2m + \frac{3}{2}$, $\nu = 2n + \frac{3}{2}$. We have†

$$J_\mu(av)J_\nu(av) = \sum_{n=0}^\infty \frac{(-1)^n(\frac{av}{2})^{\mu+\nu+2n}\,\Gamma(\mu+\nu+1+2n)}{n!\,\Gamma(\mu+1+n)\,\Gamma(\nu+1+n)\,\Gamma(\mu+\nu+1+n)}$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!}\left(\frac{av}{2}\right)^{\mu+\nu+2n} \frac{2^{\mu+\nu+1+2n}\,\Gamma\left(\frac{\mu+\nu+1}{2}+n\right)\Gamma\left(\frac{\mu+\nu+2}{2}+n\right)}{2\sqrt{\pi}\,\Gamma(\mu+1+n)\,\Gamma(\nu+1+n)\,\Gamma(\mu+\nu+1+n)}.$$

Substituting this sum in our integral and interchanging summation and integration we obtain

$$-\operatorname{Im} g^*_{m,n}(a) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{a^{\mu+\nu+2n}\,\Gamma\left(\frac{\mu+\nu+1}{2}+n\right)\Gamma\left(\frac{\mu+\nu+2}{2}+n\right)}{\Gamma(\mu+1+n)\,\Gamma(\nu+1+n)\,\Gamma(\mu+\nu+1+n)} \int_0^1 \sqrt{1-v^2}\, v^{\mu+\nu-2+2n}dv.$$

Evaluating the integral on the right, we have

$$\int_0^1 \sqrt{1-v^2}\, v^{\mu+\nu-2+2n}dv = \int_0^{\pi/2} (\sin\theta)^{2\left(\frac{\mu+\nu-2}{2}+n\right)}\cos^2\theta\, d\theta = \frac{\Gamma\left(\frac{\mu+\nu-1}{2}+n\right)\Gamma(\frac{3}{2})}{2\,\Gamma\left(\frac{\mu+\nu+2}{2}+n\right)}.$$

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† see [6], p. 147.

So, finally,

$$-\operatorname{Im} g^*_{m,n}(\alpha) = \frac{\alpha^{\mu+\nu}}{4} \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}+n\right)\Gamma\left(\frac{\mu+\nu+1}{2}+n\right)}{\Gamma(\mu+1+n)\Gamma(\nu+1+n)\Gamma(\mu+\nu+1+n)} = \frac{\alpha^{\mu+\nu}}{4} \sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!} f(n),$$

where

$$f(n) = \frac{\Gamma\left(\frac{\mu+\nu+1}{2}+n\right)\Gamma\left(\frac{\mu+\nu-1}{2}+n\right)}{\Gamma(\mu+1+n)\Gamma(\nu+1+n)\Gamma(\mu+\nu+1+n)} .$$

To obtain the asymptotic expansion of a series of the form

$$\sum_{n=0}^{\infty} \frac{(-\alpha^2)^n}{n!} f(n) ,$$

for large positive α we use a method due to E.M. Wright [7].

Using Wright's notation we now state that the following asymptotic expansion holds:

$$-\operatorname{Im} g^*_{m,n}(\alpha) \sim I(Z_1) + I(Z_2) + J(\alpha^2) ,$$

where the meaning of these symbols is defined for our case by (for details see [7]):

$$I(x) = x^{\theta} e^{x} \left\{ \sum_{m=0}^{M-1} A_m x^{-m} + O(x^{-M}) \right\}$$

$$J(y) = \sum_{r=1}^{p} \sum_{\ell=0}^{L_r} P_{r,\ell}\, y^{-(\ell+\beta_r)/\alpha_r} + O\left(y^{-N+\delta}\right) .$$

In the above equations M denotes any positive integer; N is an integer which may be taken as large as we please; and δ is an arbitrary positive number, and finally L_r is an arbitrary integer. If

$$f(t) = \frac{\Gamma(\beta_1+\alpha_1 t)\,\Gamma(\beta_2+\alpha_2 t)}{\Gamma(\mu_1+\rho_1 t)\Gamma(\mu_2+\rho_2 t)\Gamma(\mu_3+\rho_3 t)}$$

has a pole of order s at the point $-(\ell+\beta_r)/\alpha_r$, then s $P_{r,\ell}\ y^{-(\ell+\beta_r)/\alpha_r}$ is the residue of $\Gamma(-t)f(t)y^t$ at this point; if a > 1, then $P_{r,\ell}$ is a polynomial in log y of degree s - 1. We know that

$$\Gamma(\beta+t\alpha) = e^{-\alpha t}(\alpha t)^{\beta+\alpha t-1/2}\left\{\sum_{m=0}^{M-1} a_m t^{-m} + O(t^{-M})\right\} .$$

Using this fact we arrive at the following inequality:

$$\left| \frac{f(t)}{4^t\Gamma(t+1)} - \sum_{m=0}^{M-1} \frac{A_m}{\Gamma(2t-\theta+m+1)} \right| < \frac{K}{|\Gamma(2t-\theta+M+1)|} ,$$

where K is independent of t.

This inequality serves to determine the number A_o, A_1, ... uniquely. In particular, if we let M = 1 and let $t \to \infty$, we get

$$A_o = 2^{2m+2n+6}/\sqrt{2\pi} .$$

Using this procedure we can define further A_i's recursively.

Finally, we need the following:

$$p = 2, \quad q = 3, \quad \alpha_1 = \alpha_2 = \rho_1 = \rho_2 = \rho_3 = 1$$

$$\beta_1 = \frac{\mu+\nu-1}{2}, \qquad \beta_2 = \frac{\mu+\nu+1}{2}$$

$$\mu_1 = \mu+1, \quad \mu_2 = \nu+1, \quad \mu_3 = \mu+\nu+1$$

$$\theta = \sum_{r=1}^{p} \beta_r - \sum_{r=1}^{q} \mu_r + \frac{q-p}{2} = -2m-2n - \frac{11}{2}$$

$$Z_1 = 2\alpha i, \qquad Z_2 = -2\alpha i .$$

Now f(t) has a simple pole at $t = -\beta_1/\alpha_1 = -(\mu+\nu-1)/2$. All its other poles are of order 2 and occur at

$$t = -\left(\frac{\beta_1+\ell}{\alpha_1}\right) = -\left(\frac{\mu+\nu-1+2\ell}{2}\right) \qquad \text{for} \qquad \ell = 1,2, \dots .$$

The residue of $\Gamma(-t)f(t)y^t$ at

$$t = -\frac{\mu+\nu-1}{2}$$

is

$$\frac{\Gamma(\frac{\mu+\nu-1}{2})\; y^{-(\mu+\nu-1)/2}}{\Gamma(\frac{\mu+\nu+3}{2})\,\Gamma(\frac{\mu-\nu+3}{2})\,\Gamma(\frac{\nu-\mu+3}{2})} .$$

The residue of $Q(t) = \Gamma(-t)f(t)y^t$ at

$$t = -\left(\frac{\mu+\nu+2\ell-1}{2}\right) = -(\beta_1+\ell)$$

is obtained as follows. We first obtain the Laurent expansion of

$$\Gamma(\beta_1+\alpha_1 t) = \Gamma(\beta_1+t), \quad \text{at} \quad t = -(\beta_1+\ell) .$$

Let $Z_1 = \beta_1 + t$. Then $Z_1 = -\ell$ when $t = -(\beta_1+\ell)$, and we now need the Laurent expansion of $\Gamma(Z_1)$ about $Z_1 = -\ell$. We obtain (cf. [8], p. 46)

$$\Gamma(Z_1) = \frac{(-1)^\ell}{\ell!}\left\{(Z_1+\ell)^{-1} + \Psi(\ell+1) + \dots\right\} .$$

Next we need the Laurent expansion of

$$\Gamma(\beta_2+\alpha_2 t) = \Gamma(\beta_1+1+t) \quad \text{at} \quad t = -(\beta_1+\ell) .$$

Let $Z_2 = \beta_1 + 1 + t$. Then $Z_2 = 1-\ell$ when $t = -(\beta_1+\ell)$, and now we seek the Laurent expansion of $\Gamma(Z_2)$ about $Z_2 = 1-\ell = -(\ell-1)$; we obtain

$$\Gamma(Z_2) = \frac{(-1)^{\ell-1}}{(\ell-1)!}\left\{(Z_2+\ell-1)^{-1} + \Psi(\ell) + \dots\right\} .$$

We now obtain the expansion of

$$\frac{\Gamma(-t)y^t}{\Gamma(\mu_1+t)\Gamma(\mu_2+t)\Gamma(\mu_3+t)} = P(t)$$

at the regular point $t = -(\beta_1+\ell)$:

$$P(t) = P\Big(-(\beta_1+\ell)\Big) + P'\Big(-(\beta_1+\ell)\Big)\Big(t+(\beta_1+\ell)\Big) + \cdots$$

$$= \frac{\Gamma(\beta_1+\ell)y^{-(\beta_1+\ell)}}{\Gamma(\mu_1-\beta_1-\ell)\Gamma(\mu_2-\beta_1-\ell)\Gamma(\mu_3-\beta_1-\ell)}$$

$$+ \left[\frac{d}{dt}\left\{\frac{\Gamma(-t)}{\Gamma(\mu_1+t)\Gamma(\mu_2+t)\Gamma(\mu_3+t)}\right\}\Bigg|_{t=-(\beta_1+\ell)} y^{-(\beta_1+\ell)}\right.$$

$$\left. + \frac{\Gamma(-t)}{\Gamma(\mu_1+t)\Gamma(\mu_2+t)\Gamma(\mu_3+t)}\Bigg|_{t=-(\beta_1+\ell)} \log y\; y^{-(\beta_1+\ell)}\right](t+\beta_1+\ell) .$$

Now

$$\Gamma(Z_1) = \frac{(-1)^\ell}{\ell!}\left\{(t+\beta_1+\ell)^{-1} + \Psi(\ell+1) + \cdots\right\},$$

$$\Gamma(Z_2) = \frac{(-1)^{\ell-1}}{(\ell-1)!}\left\{(t+\beta_1+\ell)^{-1} + \Psi(\ell) + \cdots\right\},$$

and

$$P(t) = y^{-(\beta_1+\ell)}\left\{\frac{\Gamma(\beta_1+\ell)}{\Gamma(\mu_1-\beta_1-\ell)\Gamma(\mu_2-\beta_1-\ell)\Gamma(\mu_3-\beta_1-\ell)}\right.$$

$$\left. + \left[\frac{d}{dt}\left(\frac{\Gamma(-t)}{\prod_{i=1}^{3}\Gamma(\mu_i+t)}\right)\Bigg|_{t=-(\beta_1+\ell)} + \frac{\Gamma(-t)}{\prod_{i=1}^{3}\Gamma(\mu_1+t)}\Bigg|_{t=-(\beta_1+\ell)} \log y\right](t+\beta_1+\ell)\right\}$$

$$= y^{-(\beta_1+\ell)}\left\{A + (B + C\log y)(t+\beta_1+\ell)\right\} ,$$

where the meaning of A, B, C is obvious. Finally

$$\Gamma(Z_1)\Gamma(Z_2) = \frac{(-1)}{\ell!(\ell-1)!}\left\{\frac{1}{(t+\beta_1+\ell)^2} + \frac{\Psi(\ell)+\Psi(\ell+1)}{(t+\beta_1+\ell)} + \cdots\right\} .$$

Hence the residue of $Q(t) = \Gamma(-t)f(t)y^t$ at $t = -(\beta_1+\ell)$ is

$$\frac{(-1)y^{-(\beta_1+\ell)}}{\ell!(\ell-1)!}\left[A\left(\Psi(\ell)+\Psi(\ell+1)\right) + (B + C\log y)\right] .$$

With these calculations before us we write the asymptotic expansion of

$$-\operatorname{Im}\left\{\overset{*}{g}_{m,n}(\alpha)\right\} = \sum_{n=0}^{\infty}\frac{(-\alpha^2)^n}{n!}f(n)$$

for large positive α. It is given by

$$\begin{aligned} I(Z_1) + I(Z_2) + J(\alpha^2) = \frac{\alpha^{\mu+\nu}}{4}&\left\{(2\alpha i)^{\Theta}e^{2\alpha i}\left[\sum_{m=0}^{M}A_m(2\alpha i)^{-m} + O\left((2\alpha i)^{-M}\right)\right]\right.\\ &\left.+ (-2\alpha i)^{\Theta}e^{-2\alpha i}\left[\sum_{m=0}^{M}A_m(-2\alpha i)^{-m} + O(\alpha^{-M})\right]\right\}\\ &+\frac{\alpha^{\mu+\nu}}{4}\left\{\frac{\Gamma\left(\frac{\mu+\nu-1}{2}\right)}{\Gamma\left(\frac{\mu+\nu+3}{2}\right)\Gamma\left(\frac{\mu-\nu+3}{2}\right)\Gamma\left(\frac{\nu-\mu+3}{2}\right)}\frac{1}{\alpha^{\mu+\nu-1}}\right.\\ &\left.+\sum_{\ell=1}^{\infty}\frac{-1}{\ell!(\ell-1)!}\frac{1}{\alpha^{\mu+\nu+2\ell-1}}\left\{\begin{aligned}&A\left(\Psi(\ell)+\Psi(\ell+1)\right)\\&+(B+2C\log\alpha)\end{aligned}\right\}\right\} . \end{aligned}$$

$\Psi(Z)$ is the logarithmic derivative of the gamma function:

$$\Psi(Z) = \frac{d}{dz}\log\Gamma(Z) = \frac{\Gamma'(Z)}{\Gamma(Z)} .$$

We now consider the real part

$$\int_1^{\infty} \frac{\sqrt{v^2-1}}{v^2} J_\mu(av)\, J_\nu(av)\, dv = \mathrm{Re}\left\{ g^*_{m,n}(a) \right\} .$$

We remark first that if only a few terms of the asymptotic expansion of the real part are desired, we can obtain these by substituting for $J_\mu(av)$ and $J_\nu(av)$ the first few terms of their known asymptotic expansions (in fact these expansions consist of a finite number of terms). In what follows we give a slight modification of this procedure in which we will again make use of Wright's methods.

We have

$$J_\mu(av) J_\nu(av) = \frac{(av)^{\mu+\nu}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-a^2v^2)^n}{n!} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}+n\right)\Gamma\left(\frac{\mu+\nu+2}{2}+n\right)}{\Gamma(\mu+1+n)\,\Gamma(\nu+1+n)\,\Gamma(\mu+\nu+1+n)} .$$

Since the range of integration is from 1 to infinity we know that av is large if a is large. We obtain now the asymptotic expansion of this product for large av. Let

$$g(t) = \frac{\Gamma\left(\frac{\mu+\nu+1}{2}+t\right)\Gamma\left(\frac{\mu+\nu+2}{2}+t\right)}{\Gamma(\mu+1+t)\,\Gamma(\nu+1+t)\,\Gamma(\mu+\nu+1+t)} .$$

The following will be needed:

$$p = 2, \quad q = 3, \quad \alpha_1 = \alpha_2 = \rho_1 = \rho_2 = \rho_3 = 1 ;$$

$$\beta_1 = \frac{\mu+\nu+2}{2}, \qquad \beta_2 = \frac{\mu+\nu+1}{2} ;$$

$$\mu_1 = \mu+1, \quad \mu_2 = \nu+1, \quad \mu_3 = \mu+\nu+1 ;$$

$$\theta = \sum_1^p \beta_r - \sum_1^q \mu_r + \frac{1}{2}(q-p) = -2m-2n-4 ;$$

$$\left| \frac{g(t)}{4^t \Gamma(t+1)} - \sum_{m=0}^{M-1} \frac{A_m}{\Gamma(2t-\theta+m+1)} \right| < \frac{K}{|\Gamma(2t-\theta+M+1)|} ,$$

where K is independent of t. This last inequality makes possible the calculation of the A_i. In particular $A_o = 1/\sqrt{\pi}\ 2^{2m+2n+4}$. Also

$$Z_1 = 2\alpha vi , \qquad Z_2 = -2\alpha vi .$$

g(t) has simple poles only and these occur at $t = -(\beta_2 + \ell)$, $\ell = 0,1,2,\ldots,(m+n+2)$.

The residue of $\Gamma(-t)g(t)y^t$ at $t = -(\beta_2+\ell)$ is

$$\frac{\Gamma(\beta_2+\ell)\frac{(-1)^\ell}{\ell!}\Gamma(\frac{1}{2}-\ell)\, y^{-(\frac{\mu+\nu+1}{2}+\ell)}}{\Gamma(\frac{\mu-\nu+1}{2}-\ell)\Gamma(\frac{\nu-\mu+1}{2}-\ell)\Gamma(\frac{\mu+\nu+1}{2}-\ell)} .$$

Thus the asymptotic expansion of $J_\mu(\alpha v)J_\nu(\alpha v)$ for αv large is

$$\frac{(\alpha v)^{\mu+\nu}}{\sqrt{\pi}} \left\{ (2\alpha vi)^\theta e^{2\alpha vi} \sum_{m=0}^{M} A_m(2\alpha vi)^{-m} + O\left((\alpha v)^{-M}\right) \right.$$

$$+ (-2\alpha vi)^\theta e^{-2\alpha vi} \sum_{m=0}^{M} A_m(-2\alpha vi)^{-m} + O\left((-\alpha v)^{-M}\right)$$

$$\left. + \sum_{\ell=0}^{m+n+2} \frac{(-1)^\ell}{\ell!} \frac{\Gamma(\frac{\mu+\nu+1}{2}+\ell)\Gamma(\frac{1}{2}-\ell)}{\Gamma(\frac{\mu-\nu+1}{2}-\ell)\Gamma(\frac{\nu-\mu+1}{2}-\ell)\Gamma(\frac{\mu+\nu+1}{2}-\ell)(\alpha v)^{\mu+\nu+1+2\ell}} \right\} .$$

We now multiply by $\frac{\sqrt{v^2-1}}{v^2}$ and integrate from 1 to infinity. Then we get essentially two different types of integrals to evaulate. The first type is of the form

$$\int_1^\infty \frac{\sqrt{v^2-1}}{v^{3+2\ell}}\,dv = \int_o^1 t^{2\ell}\sqrt{1-t^2}\,dt = \int_o^{\frac{\pi}{2}} (\cos\theta)^{2\ell}\sin^2\theta d\theta = \frac{\Gamma(\ell+\frac{1}{2})\Gamma(1+\frac{1}{2})}{2\Gamma(\ell+2)} .$$

The second type is more difficult to evaluate. It is of the form

$$\int_1^\infty \frac{\sqrt{v^2-1}}{v^2} \frac{1}{(\alpha v)^p} e^{2\alpha vi}\, dv , \qquad p \text{ an integer.}$$

We let v-1 = t and we get

$$\frac{e^{2\alpha i}}{\alpha^p} \int_1^\infty \frac{\sqrt{t(t+2)}}{(t+1)^{2+p}} e^{-(-2\alpha i)t}\, dt .$$

An application of Watson's lemma (see [6], p. 236) will give the asymptotic expansion of this last integral. We see then that, after some tedious calculations, we can obtain the complete asymptotic expansion of the diffraction integrals $g^*_{m,n}(\alpha)$. For the first terms of these expansions, we have:

Theorem 2. Let $g^*_{m,n}(\alpha)$ be the functions defined by equation (4), and let $\alpha \to \infty$. Then the asymptotic expansion of the $g^*_{m,n}(\alpha)$ up to terms of the order $\alpha^{-5/2}$ is given by

$$\operatorname{Re} g^*_{m,n}(\alpha) \sim \frac{(-1)^{m+n}}{4\alpha} + (-1)^{m+n} \frac{\sqrt{\pi}}{2} \frac{\cos\left(2\alpha + \frac{3\pi}{4}\right)}{\alpha^{5/2}} + \cdots ,$$

$$\operatorname{Im} g^*_{m,n}(\alpha) \sim - \frac{\Gamma(m+n+2)}{4\Gamma(m+n+3)\,\Gamma(m-n+3/2)\,\Gamma(n-m+3/2)}\, \alpha$$

$$+ \frac{(-1)^{n+m}}{2\pi} \frac{\log \alpha}{\alpha} + \frac{(1-2\gamma)(-1)^{n+m}}{4\pi} \frac{1}{\alpha}$$

$$+ \frac{B(t)}{4} \frac{1}{\alpha} + \frac{(-1)^{m+n+1}}{2\sqrt{\pi}} \frac{\cos\left(2\alpha - \frac{3\pi}{4}\right)}{\alpha^{5/2}} + \cdots ,$$

where γ denotes Euler's constant and where

$$B(t) = \frac{d}{dt} \left\{ \frac{\Gamma(-t)}{\Gamma(2m+5/2+t)\,\Gamma(2n+5/2+t)\,\Gamma(2m+2n+4+t)} \right\}$$

evaluated at $t = -(m+n+2)$.

4. An approximation formula for large α

In this section we study the next term in the asymptotic expansion for $\alpha \to \infty$ of the solution $\mathcal{E}(x)$ of our integral equation

$$(9) \qquad -\frac{i\alpha y}{\sqrt{\pi}} = \mathcal{E}(y) + \frac{2\alpha}{\pi} \int_0^1 G(x,y;\alpha)\, \mathcal{E}(x)\, dx .$$

To obtain this information, we integrate (9). We get

$$\int_0^z -\frac{i\alpha y}{\sqrt{\pi}}\, dy = \int_0^z \mathcal{E}(y) dy + \frac{2\alpha}{\pi} \int_0^z \int_0^1 G(x,y;\alpha)\, \mathcal{E}(x) dx dy$$

or

$$(15a) \qquad -\frac{i\alpha z^2}{2\sqrt{\pi}} = E(z) + \frac{2\alpha}{\pi} \int_0^1 K(z,x)\, E'(x)\, dx ,$$

where

$$0 \leq z \leq 1 , \qquad E(z) = \int_0^z \mathcal{E}(y)\, dy$$

and

$$K(z,x) = \int_0^z G(y,x) dy = -\frac{i}{2} \int_0^1 \frac{\sqrt{1-w^2}}{i\alpha w^2} \Big[2e^{i\alpha wx} - e^{i\alpha w(x+z)} - 1 + \operatorname{sgn}(z-x)(e^{i\alpha w|z-x|} - 1) \Big]\, dw .$$

For later use we note

$$\frac{\partial K}{\partial x} = -\frac{i}{2} \int_0^1 \frac{\sqrt{1-w^2}}{w} \left\{ 2e^{i\alpha wx} - e^{i\alpha w(x+z)} - e^{i\alpha w|x-z|} \right\} dw .$$

An integration by parts of (15a) gives

$$(15b) \qquad -\frac{i\alpha z^2}{2\sqrt{\pi}} = E(z) + \frac{2\alpha}{\pi} \left[K(z,1)E(1) - \int_0^1 \frac{\partial K}{\partial x} E(x) dx \right] ,$$

where

$$K(z,\sigma) = \int_0^z G(\sigma,y;\alpha)\, dy = 0$$

since
$$G(\sigma,y;\alpha) = -\frac{\pi}{4}\int_{|\sigma-y|}^{|\sigma+y|} \frac{J_1(\alpha\tau)+iH_1(\alpha\tau)}{\tau}\,d\tau \equiv 0 .$$

Equation (15b) is exact. The limiting solution as $\alpha\to\infty$ will therefore be given by
$$E_o(z) = \int_o^z \frac{1}{\sqrt{\pi}}\,\mathcal{E}_o(x)\,dx = -\frac{1}{\sqrt{\pi}}\sqrt{1-z^2} + \frac{1}{\sqrt{\pi}} .$$

We wish now to obtain some information about the next term $E_1(z)$ in the asymptotic expansion of $E(z)$. Towards this end we now make the following approximations in equation (15b). We write $E(z) = E_o(z) + E_1(z) + \ldots$ and substitute this into (15b) dropping terms of higher order. We therefore replace $E(z)$ by $E_o(z)$, $E(1)$ by $E_o(1)$ and $E(x)$ (which occurs in the integrand) by $E_o(x) + E_1(x)$. We get

$$-\frac{i\alpha z^2}{2\sqrt{\pi}} - E_o(z) - \frac{2\alpha}{\pi}K(z,1)\frac{1}{\sqrt{\pi}} + \frac{2\alpha}{\pi}\int_o^1 \frac{\partial K}{\partial x}E_o(x)\,dx$$

$$(16) \qquad = -\frac{i\alpha z^2}{2\sqrt{\pi}} - E_o(z) - \frac{2\alpha}{\pi^{3/2}}K(z,1) + \frac{2\alpha}{\pi^{3/2}}\int_o^1 \frac{\partial K}{\partial x}\,dx$$

$$-\frac{2\alpha}{\pi^{3/2}}\int_o^1 \frac{\partial K}{\partial x}\sqrt{1-x^2}\,dx = -\frac{2\alpha}{\pi}\int_o^1 \frac{\partial K}{\partial x}E_1(x)\,dx .$$

This is an integral equation of the first kind for the determination of $E_1(x)$, since the left-hand side contains known quantities.

We now show that after division by α the left-hand side of (16) vanishes to a higher order than $\alpha^{-5/2}$ as $\alpha\to\infty$. For this purpose we need the following calculations:

1) $-\dfrac{2\alpha}{\pi\sqrt{\pi}} K(z,1)$:

$$K(z,1) = -\frac{i}{2}\frac{1}{i\alpha}\int_0^1 \frac{\sqrt{1-w^2}}{w^2}\left\{2e^{i\alpha w} - e^{i\alpha w(1+z)} - e^{i\alpha w(1-z)}\right\} dw$$

$$= -\frac{1}{2\alpha}\int_0^1 \frac{\sqrt{1-w^2}}{w^2}\left\{2\cos\alpha w - \cos\alpha w(1+z) - \cos\alpha w(1-z)\right\} dw$$

$$-\frac{i}{2\alpha}\int_0^1 \frac{\sqrt{1-w^2}}{w^2}\left\{2\sin\alpha w - \sin\alpha w(1+z) - \sin\alpha w(1-z)\right\} dw$$

$$= -\frac{1}{2\alpha}\int_0^1 dw\, \frac{\sqrt{1-w^2}}{w^2}\left\{\left(+2\sin(\alpha w + \frac{\alpha wz}{2})\sin\frac{\alpha wz}{2}\right) - \left(2\sin(\alpha w - \frac{\alpha wz}{2})\sin\frac{\alpha wz}{2}\right)\right\}$$

$$-\frac{i}{2\alpha}\int_0^1 dw\, \frac{\sqrt{1-w^2}}{w^2}\left\{\left(-2\cos(\alpha w + \frac{\alpha wz}{2})\sin\frac{\alpha wz}{2}\right) + \left(2\cos(\alpha w - \frac{\alpha wz}{2})\sin\frac{\alpha wz}{2}\right)\right\}$$

$$= -\frac{2^2}{2\alpha}\int_0^1 dw\, \frac{\sqrt{1-w^2}}{w^2}\sin\frac{\alpha wz}{2}\cos\alpha w\sin\frac{\alpha wz}{2}$$

$$+\frac{i2^2(-1)}{2\alpha}\int_0^1 dw\, \frac{\sqrt{1-w^2}}{w^2}\sin\frac{\alpha wz}{2}\sin\alpha w\sin(\frac{\alpha wz}{2})$$

$$\mp\frac{4i}{2\alpha}\int_0^1 \sin^2\frac{\alpha wz}{2}\sin\alpha w\,\frac{\sqrt{1-w^2}}{w^2}\, dw$$

$$\mp\frac{4}{2\alpha}\int_0^1 \sin^2\frac{\alpha wz}{2}\cos\alpha w\,\frac{\sqrt{1-w^2}}{w^2}\, dw\,.$$

Therefore

$$-\frac{2\alpha}{\pi\sqrt{\pi}}K(z,1) = \frac{4}{\pi\sqrt{\pi}}\left\{\int_0^1 \sin^2\frac{\alpha wz}{2}\cos\alpha w\,\frac{\sqrt{1-w^2}}{w^2}\,dw + i\int_0^1 \sin^2\frac{\alpha wz}{2}\sin\alpha w\,\frac{\sqrt{1-w^2}}{w^2}\,dw\right\}.$$

Now we need the value of $\frac{2\alpha}{\pi^{3/2}} \int_0^1 \frac{\partial K}{\partial x} dx$. Calculations 2) and 3) give its value:

2)
$$\frac{2\alpha}{\pi} \left(-\frac{i}{2}\right)\left(\div \frac{1}{\sqrt{\pi}}\right) \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w} \int_0^1 \Big(2 \cos \alpha wx - \cos \alpha w(z+x) - \cos \alpha w(x-z)\Big) \; dx$$

$$- \frac{i\alpha}{\pi^{3/2}} \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w} \left\{ \left(\frac{2 \sin \alpha wx}{\alpha w} - \frac{\sin \alpha w(z+x)}{\alpha w} - \frac{\sin \alpha w(x-z)}{+\alpha w}\right) \Big|_0^1 \right\}$$

$$= - \frac{i}{\pi^{3/2}} \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w^2} \left\{ 2 \sin \alpha w - \sin \alpha w(z+1) - \sin \alpha w(1-z) \right\}$$

$$= - \frac{i}{\pi^{3/2}} \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w^2} \left\{ 2 \sin \alpha w - 2 \sin \alpha w \cos \alpha wz \right\}$$

$$= - \frac{2i}{\pi^{3/2}} \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w^2} \left\{ 1 - \cos \alpha wz \right\} \sin \alpha w$$

$$= - \frac{4i}{\pi^{3/2}} \int_0^1 \frac{\sqrt{1-w^2}}{w^2} \sin \alpha w \sin^2 \frac{\alpha wz}{2} \; dw \; ;$$

3)
$$\frac{2\alpha}{\pi} \left(-\frac{i}{2}\right)\left(+\frac{1}{\sqrt{\pi}}\right) i \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w} \int_0^1 \Big(2 \sin \alpha wx - \sin \alpha w(z+x) - \sin \alpha w|z-x|\Big) \; dx$$

$$= \frac{\alpha}{\pi \sqrt{\pi}} \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w} \left\{ - 2 \frac{\cos \alpha wx}{\alpha w} \Big|_0^1 + \frac{\cos \alpha w(z+x)}{\alpha w} \Big|_0^1 \right.$$

$$\left. + \frac{\cos \alpha w(z-x)}{- \alpha w} \Big|_0^z + \frac{\cos \alpha w(x-z)}{\alpha w} \Big|_z^1 \right\}$$

$$= \frac{1}{\pi\sqrt{\pi}} \int_0^1 dw \frac{\sqrt{1-w^2}}{w^2} \begin{Bmatrix} -2\cos\alpha w + 2 + \cos\alpha w(z+1) - \cos\alpha wz \\ -1 + \cos\alpha wz + \cos\alpha w(1-z) - 1 \end{Bmatrix}$$

$$= \frac{1}{\pi\sqrt{\pi}} \int_0^1 dw \frac{\sqrt{1-w^2}}{w^2} \left\{ -2\cos\alpha w + 2\cos\alpha w \cos\alpha wz \right\}$$

$$= -\frac{2}{\pi\sqrt{\pi}} \int_0^1 dw \frac{\sqrt{1-w^2}}{w^2} \left\{ 1 - \cos\alpha wz \right\} \cos\alpha w$$

$$= -\frac{4}{\pi\sqrt{\pi}} \int_0^1 dw \frac{\sqrt{1-w^2}}{w^2} \sin^2\frac{\alpha wz}{2} \cos\alpha w \; .$$

We notethat 1) + 2) + 3) = 0. Consequently equation (16) simplies to

$$\text{(16a)} \qquad \frac{\sqrt{\pi}\, iz^2}{4} + \frac{\pi}{2\alpha} E_o(z) + \frac{1}{\sqrt{\pi}} \int_0^1 \frac{\partial K}{\partial x} \sqrt{1-x^2}\, dx = \int_0^1 \frac{\partial K}{\partial x} E_1(x) dx \; .$$

We consider now the integral in the left-hand side of equation (16a):

$$\frac{1}{\sqrt{\pi}} \int_0^1 \frac{\partial K}{\partial x} \sqrt{1-x^2}\, dx = -\frac{i}{2\sqrt{\pi}} \int_0^1 dw \frac{\sqrt{1-w^2}}{w} \int_0^1 dx \sqrt{1-x^2} \Big\{ \Big(2\cos\alpha wx - \cos\alpha w(z+x) -$$

$$- \cos\alpha w(z-x) \Big) + i\Big(2\sin\alpha wx - \sin\alpha w(z+x) - \sin\alpha w|z-x| \Big) \Big\} .$$

We wish to know the value of the imaginary part of this last expression as $\alpha \to \infty$. We have

$$-\frac{i}{2\sqrt{\pi}}\int_0^1 dw\,\frac{\sqrt{1-w^2}}{w}\int_0^1\sqrt{1-x^2}\,\Big(2\cos\alpha wx-\cos\alpha w(z+x)-\cos\alpha w(z-x)\Big)\,dx$$

$$=-\frac{i}{2\sqrt{\pi}}\int_0^1 dw\,\frac{\sqrt{1-w^2}}{w}\,4\sin^2\frac{\alpha wz}{2}\int_0^1\cos\alpha wx\,\sqrt{1-x^2}\,dx$$

$$=-i\sqrt{\pi}\int_0^1 dw\,\frac{\sqrt{1-w^2}}{\alpha w^2}\,J_1(\alpha w)\sin^2\frac{\alpha wz}{2}$$

$$=-i\sqrt{\pi}\int_0^\alpha \sin^2\frac{tz}{2}\,\frac{J_1(t)}{t^2}\,\sqrt{1-\left(\frac{t}{\alpha}\right)^2}\,dt\,.$$

The limit as $\alpha\to\infty$ of this last expression is

$$-i\sqrt{\pi}\int_0^\infty \sin^2\frac{tz}{2}\,\frac{J_1(t)}{t^2}\,dt = R(z)\,,$$

and

$$R'(z) = -\frac{i\sqrt{\pi}}{2}\int_0^\infty \sin tz\,\frac{J_1(t)}{t}\,dt = -\frac{i\sqrt{\pi}}{2}\,z$$

(cf. [4], p. 36). Hence

$$R(z) = -\frac{i\sqrt{\pi}\,z^2}{4}$$

and this cancels the first term of the left-hand side of equation (16a).

We now differentiate equation (16a) with respect to α, and evaluate $\frac{\partial}{\partial\alpha}\int_0^1\frac{\partial K}{\partial x}\sqrt{1-x^2}\,dx$. We have, considering each term separately,

$$A(\alpha) = 2i\left\{\int_0^1\int_0^1 x\cos\alpha wx\sqrt{1-w^2}\sqrt{1-x^2}\,dwdx + i\int_0^1\int_0^1 x\sin\alpha wx\sqrt{1-w^2}\sqrt{1-x^2}\,dwdx\right\};$$

$$A_1(\alpha) = 2i\,\frac{\pi}{2}\int_0^1 \frac{J_1(\alpha x)}{\alpha}\sqrt{1-x^2}\,dx = \frac{-\pi i}{\alpha}\int_0^1 J_0'(\alpha x)\sqrt{1-x^2}\,dx$$

$$= \frac{-\pi i}{\alpha}\left\{\left(\frac{J_0(\alpha x)}{\alpha}\sqrt{1-x^2}\right)\Big|_0^1 + \frac{1}{\alpha}\int_0^1 J_0(\alpha x)\frac{x}{\sqrt{1-x^2}}\,dx\right\}$$

$$= \frac{-\pi i}{\alpha}\left\{-\frac{1}{\alpha} + \frac{1}{\alpha}\int_0^1 J_0(\alpha x)\frac{x}{\sqrt{1-x^2}}\,dx\right\};$$

$$A_2(\alpha) = -2\,\frac{\pi}{2\alpha}\int_0^1 H_1(\alpha x)\sqrt{1-x^2}\,dx = \frac{-\pi}{\alpha}\left\{\int_0^1 \frac{2}{\pi}\sqrt{1-x^2}\,dx - \int_0^1 H_0'(\alpha x)\sqrt{1-x^2}\,dx\right\}$$

$$= -\frac{2}{\alpha}\frac{\pi}{4} + \frac{\pi}{\alpha}\left\{\left(\frac{H_0(\alpha x)}{\alpha}\sqrt{1-x^2}\right)\Big|_0^1 + \frac{1}{\alpha}\int_0^1 H_0(\alpha x)\frac{x}{\sqrt{1-x^2}}\,dx\right\};$$

$$B(\alpha) = -i\int_0^1\int_0^1 (x+z)e^{i\alpha w(x+z)}\sqrt{1-w^2}\sqrt{1-x^2}\,dxdw$$

$$= -i\int_0^1\int_0^1 (x+z)\left\{(\cos\alpha w(x+z) + i\sin\alpha w(x+z)\right\}\sqrt{1-w^2}\sqrt{1-x^2}\,dxdw;$$

$$B_1 = -i\int_0^1\int_0^1 (x+z)\cos\alpha w(x+z)\sqrt{1-w^2}\sqrt{1-x^2}\,dxdw$$

$$= -\frac{\pi i}{2}\int_0^1 \frac{J_1\big(\alpha(x+z)\big)}{\alpha}\sqrt{1-x^2}\,dx$$

$$= +\frac{\pi i}{2\alpha}\left\{\left(\frac{J_0\big(\alpha(x+z)\big)}{\alpha}\sqrt{1-x^2}\right)\Big|_0^1 + \frac{1}{\alpha}\int_0^1 J_0\big(\alpha(x+z)\big)\frac{x}{\sqrt{1-x^2}}\,dx\right\} \sim -\frac{\pi i}{2\alpha^2}J_0(\alpha z);$$

$$B_2(\alpha) = + \int_0^1 \int_0^1 (x+z) \sin \alpha w(x+z) \sqrt{1-w^2} \sqrt{1-x^2} \, dw dx = + \frac{\pi}{2} \int_0^1 \frac{H_1(\alpha(x+z))}{\alpha} \sqrt{1-x^2} \, dx$$

$$= + \frac{\pi}{2\alpha} \left\{ \int_0^1 \frac{2}{\pi} \sqrt{1-x^2} \, dx - \int_0^1 H_o'(\alpha(x+z)) \sqrt{1-x^2} \, dx \right\}$$

$$= + \frac{\pi}{4\alpha} - \frac{\pi}{2\alpha} \left\{ \left(\frac{H_o(\alpha(x+z))}{\alpha} \sqrt{1-x^2} \right) \Big|_o^1 + \frac{1}{\alpha} \int_o^1 H_o(\alpha(x+z)) \frac{x}{\sqrt{1-x^2}} \, dx \right\}$$

$$= + \frac{\pi}{4\alpha} + \frac{\pi}{2\alpha^2} H_o(\alpha z) ;$$

$$C(\alpha) = - i \int_o^1 \int_o^1 |x-z| \left\{ \cos \alpha w|x-z| + i \sin \alpha w|x-z| \right\} \sqrt{1-w^2} \sqrt{1-x^2} \, dw dx$$

$$= - i \frac{\pi}{2} \int_o^1 \frac{J_1(\alpha|x-z|)}{\alpha} \sqrt{1-x^2} \, dx + \frac{\pi}{2} \int_o^1 \frac{H_1(\alpha|x-z|)}{\alpha} \sqrt{1-x^2} \, dx ;$$

$$C_1(\alpha) = - \frac{\pi i}{2\alpha} \left[\int_o^z J_1(\alpha(z-x)) \sqrt{1-x^2} \, dx + \int_z^1 J_1(\alpha(x-z)) \sqrt{1-x^2} \, dx \right]$$

$$= \frac{\pi i}{2\alpha} \left[\int_o^z J_o'(\alpha(z-x)) \sqrt{1-x^2} \, dx + \int_z^1 J_o'(\alpha(x-z)) \sqrt{1-x^2} \, dx \right]$$

$$= \frac{\pi i}{2\alpha} \left\{ \left(\frac{J_o(\alpha(z-x))}{-\alpha} \sqrt{1-x^2} \right) \Big|_o^z - \frac{1}{\alpha} \int_o^z J_o(\alpha(z-x)) \frac{x}{\sqrt{1-x^2}} \, dx \right.$$

$$\left. + \frac{J_o(\alpha(x-z))}{\alpha} \sqrt{1-x^2} \Big|_z^1 + \frac{1}{\alpha} \int_z^1 J_o(\alpha(x-z)) \frac{x}{\sqrt{1-x^2}} \, dx \right\}$$

$$\sim \frac{\pi i}{2\alpha} \left\{ - \frac{\sqrt{1-z^2}}{\alpha} + \frac{J_o(\alpha z)}{\alpha} - \frac{\sqrt{1-z^2}}{\alpha} \right\} ;$$

$$C_2(\alpha) = \frac{\pi}{2\alpha}\left[\int_0^1 \frac{2}{\pi}\sqrt{1-x^2}\,dx - \int_0^1 H_0'(\alpha|x-z|)\sqrt{1-x^2}\,dx\right]$$

$$= \frac{\pi}{4\alpha} - \frac{\pi}{2\alpha}\int_0^1 H_0'(\alpha|x-z|)\sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{4\alpha} - \frac{\pi}{2\alpha}\left\{\int_0^z H_0'(\alpha(z-x))\sqrt{1-x^2}\,dx + \int_z^1 H_0'(\alpha(x-z))\sqrt{1-x^2}\,dx\right\}$$

$$= \frac{\pi}{4\alpha} - \frac{\pi}{2\alpha}\left\{\left(\frac{H_0(\alpha(z-x))}{-\alpha}\sqrt{1-x^2}\right)\Bigg|_0^z - \frac{1}{\alpha}\int_0^z H_0(\alpha(z-x))\frac{x}{\sqrt{1-x^2}}\,dx\right.$$

$$\left. + \left(\frac{H_0(\alpha(x-z))}{\alpha}\sqrt{1-x^2}\right)\Bigg|_z^1 + \frac{1}{\alpha}\int_z^1 H_0(\alpha(x-z))\frac{x}{\sqrt{1-x^2}}\,dx\right\}$$

$$\sim \frac{\pi}{4\alpha} - \frac{\pi}{2\alpha}\left\{\frac{H_0(\alpha z)}{\alpha}\right\} .$$

Combining, we find that

$$-\frac{i}{2\sqrt{\pi}}\left\{A + B + C\right\} = \frac{\sqrt{\pi}}{2\alpha^2} - \frac{\sqrt{\pi}}{2\alpha^2}\sqrt{1-z^2} + R ,$$

where

$$R = -\frac{i}{2\sqrt{\pi}}\left\{\frac{-\pi i}{\alpha^2}\int_0^1 J_0(\alpha x)\frac{x}{\sqrt{1-x^2}}\,dx\right.$$

$$+ \frac{\pi}{\alpha^2}\int_0^1 H_0(\alpha x)\frac{x}{\sqrt{1-x^2}}\,dx + \frac{\pi i}{2\alpha^2}\int_0^1 J_0(\alpha(x+z))\frac{x}{\sqrt{1-x^2}}\,dx$$

(17)

$$- \frac{\pi}{2\alpha^2}\int_0^1 H_0(\alpha(x+z))\frac{x}{\sqrt{1-x^2}}\,dx - \frac{\pi i}{2\alpha^2}\int_0^z J_0(\alpha(z-x))\frac{x}{\sqrt{1-x^2}}\,dx$$

$$\left. + \frac{\pi i}{2\alpha^2}\int_z^1 J_0(\alpha(x-z))\frac{x}{\sqrt{1-x^2}}\,dx + \frac{\pi}{2\alpha^2}\int_0^z H_0(\alpha(z-x))\frac{x}{\sqrt{1-x^2}}\,dx\right\} .$$

We now differentiate the following with respect to α:

$$q(\alpha) = \frac{\pi}{2\alpha} E_o(z) = \frac{\pi}{2\alpha} \left(-\frac{1}{\sqrt{\pi}} \sqrt{1-z^2} + \frac{1}{\sqrt{\pi}} \right) .$$

We get

$$q'(\alpha) = + \frac{\sqrt{\pi}}{2\alpha^2} \sqrt{1-z^2} - \frac{\sqrt{\pi}}{2\alpha^2} .$$

Adding, we get

$$-\frac{i}{2\sqrt{\pi}} (A + B + C) + q'(\alpha) = R .$$

When z is not in the neighborhood of 0 or 1, $R(z;\alpha)$ vanishes at least to order $\alpha^{-5/2}$ as $\alpha \to \infty$. Therefore

$$(18) \qquad T(z;\alpha) = \int_{\infty}^{\alpha} R(z;\beta)\, d\beta$$

exists. We see then that the left-hand side of equation (16a) vanishes to a higher order than $\alpha^{-5/2}$ as $\alpha \to \infty$.

We now give the value of E_1 in closed form. We have

$$\frac{\partial K}{\partial x} = -\frac{i}{2} \int_0^1 \frac{\sqrt{1-w^2}}{w} \left\{ e^{i\alpha wx} - e^{i\alpha w(z+x)} + e^{i\alpha wx} - e^{i\alpha w|z-x|} \right\} dw .$$

We have already shown that

$$\lim_{\alpha \to \infty} G(x,y;\alpha) = -\frac{i}{2} \log \left| \frac{x+y}{x-y} \right| .$$

Magnus [3] has shown that $G(x,y;\alpha)$ is also given by

$$-\frac{i}{2} \int_0^1 \frac{\sqrt{1-w^2}}{w} \left\{ e^{i\alpha w|x-y|} - e^{i\alpha w|x+y|} \right\} dw .$$

It follows that

$$\lim_{\alpha \to \infty} \frac{\partial K}{\partial x} = -\frac{i}{2} \log \left| \frac{z^2 - x^2}{x^2} \right| .$$

From the above calculations we now get

$$(19) \qquad T(z;\alpha) = -\frac{i}{2} \int_0^1 \log \left| \frac{z^2 - x^2}{x^2} \right| E_1(x)\, dx .$$

Now $T(0;\alpha) = 0$, and so we can write the above as follows:

$$(20) \quad T(z;\alpha) - T(0;\alpha) = \left(-\frac{i}{2} \int_0^1 \log\left|z^2 - x^2\right| E_1(x) dx \right) - \left(-\frac{i}{2} \int_0^1 \log x^2\, E_1(x)\, dx \right) .$$

If we find a solution for

$$(21) \qquad T(z;\alpha) = -\frac{i}{2} \int_0^1 \log \left|z^2 - x^2\right| E_1(x)\, dx$$

which is valid for $0 \leq z \leq 1$ then this solution will also satisfy (20).

We obtain a solution of (21) as follows. Let $z^2 = s$ and let $x^2 = t$.

Equation (21) becomes

$$(21a) \qquad T(\sqrt{s}\,;\alpha) = -\frac{i}{2} \int_0^1 \log|s-t|\, E_1(\sqrt{t}\,) \frac{dt}{2\sqrt{t}} .$$

Let

$$\frac{E_1(\sqrt{t}\,)}{2\sqrt{t}} = Y(t) ,$$

and we get

$$(22) \qquad 2iT(\sqrt{s}\,;\alpha) = \int_0^1 \log|s-t|\, Y(t)\, dt .$$

This last equation has been studied by T. Carleman (cf. Schmeidler [9], p. 56). The solution is given as follows

$$Y(t) = \frac{E_1(\sqrt{t})}{2\sqrt{t}} = -\frac{1}{2\pi^2 \log 2 \sqrt{t}\sqrt{1-t}} \int_0^1 \frac{T(\sqrt{s})}{\sqrt{s(1-s)}}\, ds \tag{23}$$

$$+ \frac{1}{\pi^2 \sqrt{t(1-t)}} \int_0^1 \frac{\frac{d}{ds} T(\sqrt{s})\sqrt{s(1-s)}}{s-t}\, ds$$

where the last integral is a principle value integral.

We find from (23):

$$E_1(x) = -\frac{2}{\pi^2 \log 2 \sqrt{1-x^2}} \int_0^1 \frac{T(z)}{\sqrt{1-z^2}}\, dz \tag{24}$$

$$+ \frac{2}{\pi^2 \sqrt{1-x^2}} \int_0^1 \frac{\frac{d}{dz} T(z)\sqrt{1-z^2}}{z^2-x^2}\, dz ,$$

$$\mathcal{E}_1(x) = \frac{dE_1(x)}{dx} \quad . \tag{25}$$

We state this last as

<u>Theorem 3.</u> Let the function $\mathcal{E}_1(x)$ be defined by (17), (18), (24), (25). Then $\mathcal{E}_1(x)$ is the first-order term arising from a perturbation of the integral equation (9) for $\mathcal{E}(x)$ at $\alpha = \infty$.

5. A related result

We give finally a result which, although it is not directly connected with the diffraction problem, follows immediately from it.

One of the integrals which arose in the above calculation was

$$\int_0^1 \int_0^1 \sqrt{1-x^2} \; \frac{\sqrt{1-w^2}}{w} \; \sin \alpha w(x+z) \; dw \; dx \quad .$$

We shall evaluate this integral as $\alpha \to \infty$ (in a manner different from that which has preceded). We have

$$\int_0^1 \int_0^1 \sqrt{1-x^2} \; \frac{\sqrt{1-w^2}}{w} \; \sin \alpha w(x+z) \; dw \; dx$$

$$= \int_0^1 \int_0^1 \sqrt{1-x^2} \; \frac{\sqrt{1-w^2}}{w} \left\{ \sin \alpha wx \cos \alpha wz + \cos \alpha wx \sin \alpha wz \right\} dw \; dx$$

$$= \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w} \cos \alpha wz \int_0^1 dx \; \sqrt{1-x^2} \; \sin \alpha wx$$

$$+ \int_0^1 dw \; \frac{\sqrt{1-w^2}}{w} \sin \alpha wz \int_0^1 dx \; \sqrt{1-x^2} \; \cos \alpha wx$$

$$= \frac{\pi}{2} \int_0^1 \frac{H_1(\alpha w)}{\alpha w^2} \cos \alpha wz \sqrt{1-w^2} \; dw + \frac{\pi}{2} \int_0^1 \frac{J_1(\alpha w)}{\alpha w^2} \sqrt{1-w^2} \sin \alpha wz \; dz$$

$$= \frac{\pi}{2} \int_0^\alpha \frac{H_1(t)}{t^2} \cos tz \sqrt{1 - (\tfrac{t}{\alpha})^2} \; dt + \frac{\pi}{2} \int_0^\alpha \frac{J_1(t)}{t^2} \sin tz \sqrt{1 - (\tfrac{t}{\alpha})^2} \; dt \; .$$

If we take the limit of this last expression as $\alpha \to \infty$ we get

$$\frac{\pi}{2} \left\{ \int_0^\infty \frac{H_1(t)}{t^2} \cos tz \; dt + \int_0^\infty \frac{J_1(t)}{t^2} \sin tz \; dt \right\} \quad .$$

On the other hand we have

$$\lim_{\alpha \to \infty} \int_0^1 \int_0^1 \sqrt{1-x^2} \; \frac{\sqrt{1-w^2}}{w} \sin \alpha w(x+z) \, dw \, dz$$

$$= \lim_{\alpha \to \infty} \int_0^1 \int_0^\alpha \sqrt{1-x^2} \; \sqrt{1-\left(\frac{t}{\alpha}\right)^2} \sin t(x+z) \frac{dt}{t} \; dx$$

$$= \int_0^1 dx \sqrt{1-x^2} \int_0^\infty \frac{\sin t(x+z)}{t} \, dt = \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} \, dx \; .$$

Equating these two results we get

$$\int_0^\infty \frac{H_1(t)}{t^2} \cos tz \, dt + \int_0^\infty \frac{J_1(t)}{t^2} \sin tz \, dt = \frac{\pi}{4} \; .$$

Since the sine transform of $J_1(t)/t^2$ is known we have a formula which gives the cosine transform of $H_1(t)/t^2$.

To our knowledge this has not been evaluated previously. We remark that there is some generality in the method and that other transforms can be obtained similarly. For example if $\sqrt{1-x^2}$ is replaced by $(1-x^2)^{3/2}$, etc., we can obtain in this manner

$$\left\{ \int_0^\infty \frac{H_2(t)}{t^3} \cos tz \, dt + \int_0^\infty \frac{J_2(t)}{t^3} \sin tz \, dt \right\} = \frac{1}{4 \, \Gamma(3)} \; .$$

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